Analysis of the Migdal-Kadanoff renormalisation group approach to the dilute s-state Potts model. An alternative scheme for the percolation (s to 1) limit

LETTER TO THE EDITOR

Analysis of the Migdal–Kadanoff renormalisation group approach to the dilute s-state Potts model. An alternative scheme for the percolation (s → 1) limit

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Abstract. In the dilute s-state Potts formulation of the site–bond correlated percolation, two current treatments of the single-site interaction in the Migdal–Kadanoff renormalisation group are analysed. It is pointed out that they respect the Ising symmetry and would lead always to a critical site percolation probability $p_{Sc} \geq \frac{1}{2}$, giving a phase diagram qualitatively wrong for dimensions $d > 2$. It is found that the origin of this defect is traced to the preservation of the Ising symmetry which is absent in the percolation problem. An alternative scheme is proposed which gives the correct behaviour.

In random bond percolation (for reviews see Stauffer 1979, 1981b and Essam 1980) each bond between two neighbouring sites is active with random probability p_B. In site percolation each site is occupied with random probability p_S. In random site–bond percolation (see e.g. Hoshen *et al* 1979, Agrawal *et al* 1979, Nakanishi and Reynolds 1979) bonds may only be active between occupied sites. If the sites are not randomly occupied but correlated as sites in a lattice gas model and the bonds are randomly active between occupied sites, we have the site–bond correlated percolation (Coniglio *et al* 1979, 1982).

This was originally introduced as a model for reversible gelation, but subsequently it also provided a useful scheme to describe critical 'droplets' near the lattice gas critical points (Coniglio and Klein 1980). This model has also been investigated by ε expansion (Coniglio and Lubensky 1980) and Monte Carlo methods (Stauffer 1981a, Ottavi 1981, Roussenq 1981, Hermann and Stauffer 1981).

It has been shown (Murata 1979, Coniglio and Klein 1980) that the site–bond correlated percolation can be obtained from the dilute s-state Potts Hamiltonian $\mathscr{H}$ in the limit $s \to 1$. The form of this Hamiltonian is

$$-\beta\mathscr{H} = J \sum_{\langle ij \rangle} (\delta_{\sigma_i\sigma_j} - 1)n_in_j + K \sum_{\langle ij \rangle} n_in_j - \Delta \sum_i n_i \tag{1}$$

where every site is associated with a Potts variable $\sigma_i = 1, 2, \ldots, s$ and with a lattice gas variable $n_i = 0, 1$. K is the lattice gas nearest-neighbour interaction, Δ is the

chemical potential of the occupied sites and J is related to the bond probability p_B between two occupied sites via the relation $p_B = 1 - e^{-J}$.

Hamiltonian (1) has been investigated for other reasons by various authors for any s. In particular, vacancies play an effective role in displaying the first-order phase transition exhibited by the Potts model for higher values of s (Nienhuis *et al* 1979, 1981, 1980a, b, Berker *et al* 1980, Andelman and Berker 1981, Nauenberg and Scalapino 1980).

In most cases the approximations involved are particularly troublesome in the limit $s \to 1$. We will briefly analyse in particular the approximations involved in the Migdal–Kadanoff renormalisation group approach (MKRG) (Migdal 1976, Kadanoff 1976).

In applying the MKRG to a problem, there are two general steps: (1) 'bond' moving and (2) decimation. Symbolically we may represent these two steps as $H \to \tilde{H} \to H'$ (see figure 1). Since the approximation lies in the first step, whether one gets a reasonable result from MKRG depends crucially on the choice of which 'bonds' to move (Migdal 1976, Kadanoff 1976, Emery and Swendsen 1977, Swendsen and Zia 1979). Especially when single-site interactions are involved, their effects on the undecimated (renormalised) variables can be grossly underestimated if they or a portion thereof are not moved. The choice of what portion to move is, at this level of approximation, not unique and must be guided by physical arguments.
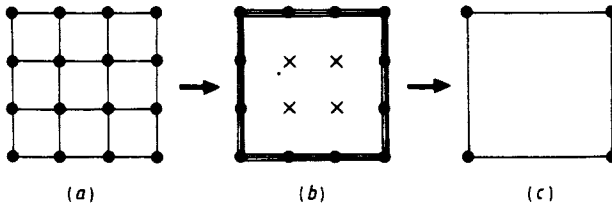
Figure 1. Successive steps of the Migdal–Kadanoff renormalisation group on the square lattice: (i) bond moving $(a) \to (b)$; (ii) decimation $(b) \to (c)$.

We briefly describe two current procedures. One is based on a suggestion due to Emery and Swendsen (1977) in which the effect of the bond moving approximation vanishes in the weak and strong limits. This is accomplished by writing Hamiltonian (1) in the following way:

$$-\beta\mathcal{H} = J \sum_{\langle ij \rangle} (\delta_{\sigma_i \sigma_j} - 1) n_i n_j - \frac{K}{2} \sum_{\langle ij \rangle} (n_i - n_j)^2 - \left(\Delta - c\frac{K}{2}\right) \sum_i n_i \qquad (2)$$

where c is the coordination number. Only the first two terms are subject to the bond moving. This procedure (Berker *et al* 1980, Andelman and Berker 1981) leads for $s = 1$ to two lines of fixed points instead of two expected fixed points (Andelman and Berker 1981).

A second procedure is based on the idea that in the strong coupling limit the density of sites must be conserved in the bond moving approximation (see e.g. Berker *et al* 1978, Coniglio and Klein 1980). This is accomplished by writing the single-site interaction in Hamiltonian (1) as a bond using the identity

$$-\Delta \sum_i n_i = -\frac{\Delta}{c} \sum_{\langle ij \rangle} (n_i + n_j). \qquad (3)$$

Moving these 'bonds' leads to (i) an invariant line $H = \frac{1}{2}(\Delta - cK/2) = 0$, which in the Ising terminology corresponds to the magnetic field $H = 0$, and (ii) two other stable lines at $\Delta = -\infty$ (all sites occupied) and $\Delta = +\infty$ (all sites empty). As a consequence, starting from any value of p_B and $H > 0$ and applying the RG, one will always reach the point $\Delta = +\infty$, where there is no percolation (all sites empty). Therefore for any given value of p_B and K the corresponding site critical probability p_{Sc} will be always greater than or equal to $\frac{1}{2}$. This may not be a serious problem for planar lattices for which $p_{Sc} \geq \frac{1}{2}$. But it gives wrong results for three-dimensional lattices for which $p_{Sc} < \frac{1}{2}$. In fact, Coniglio and Klein (1980), using the second procedure of bond moving, found for two-dimensional systems and $p_B = 1$ in the p_S–K plane a percolation line which is always above the line $p_S = \frac{1}{2}$. More precisely, this line starts at the random site percolation threshold and monotonically ends at the lattice gas critical point $K = K_c$, $p_S = \frac{1}{2}$, in agreement with rigorous (Coniglio *et al* 1977) and Monte Carlo results (Stauffer 1981a, b, Roussenq 1981).

If one did use the same procedure for three-dimensional lattices, one would find the same qualitative behaviour, namely a line of percolation points in the plane $p_S > \frac{1}{2}$ ending at the lattice critical point, contrary to the well established result (Müller-Krumbhaar 1974, Domb *et al* 1980, Hermann and Stauffer 1981) in which the percolative lines are always in the plane $p_S < \frac{1}{2}$ and end on the coexistence curve at a value $K_p > K_c$.

In order to overcome these problems we suggest an alternative scheme for moving the single-site interaction. For simplicity we restrict ourselves to the case $K = 0$ (random site–bond percolation). First consider J, the parameter in H associated with bond probability $p_B = 1 - q_B = 1 - e^{-J}$. Within the MKRG, in the first step of bond moving, we have

$$\tilde{J} = b^{d-1} J. \tag{4}$$

In terms of the physical p_B, this step corresponds to setting $p_B = 0$ ($J = 0$) on all bonds which have been moved and to balancing this loss in connectivity by increasing p_B to $\tilde{p}_B = 1 - \exp(-\tilde{J})$ on the remaining bonds (figure 1). Thus from (4)

$$\tilde{p}_B = 1 - \exp(-b^{d-1} J) \qquad \text{or} \qquad \tilde{q}_B = 1 - \tilde{p}_B = (q_b)^{b^{d-1}}. \tag{5}$$

Next we consider Δ, the parameter in $\mathcal{H}$ related to site probability $p_S = 1 - q_S$ by

$$p_S = 1/(1 + e^{\Delta}). \tag{6}$$

Motivated by the discussion associated with p_B, we choose to move a portion of Δ such that

$$\tilde{q}_S = 1 - \tilde{p}_S = (q_S)^{b^{d-1}}, \tag{7}$$

treating the site and bond probability on the same footing. In this way, the disconnection of those sites, induced by the bond moving step (shown as crosses in figure 1(*b*)), is balanced by increasing the site probability on the remaining sites (shown as dots in figure 1(*b*)). In order to achieve (7) in the bond moving, we write the single-site interaction in Hamiltonian (1) as

$$-\Delta \sum_i n_i = -(\Delta + D) \sum_i n_i + (D/c) \sum_{\langle ij \rangle} (n_i + n_j). \tag{8}$$

The D/c term is that portion of Δ which we associate with a bond and is moved along with J. Thus

$$\tilde{D} = b^{d-1}D. \tag{9}$$

The two standard procedures discussed above are recovered respectively by choosing $D = 0$ and $D = -\Delta$. Here we choose D such that (7) is satisfied, namely

$$e^{-D} = q_s = (1 + e^{-\Delta})^{-1}. \tag{10}$$

With this choice the decimation step $\tilde{H} \to H'$, for a simple hypercubic lattice in d dimensions with coordination number $c = 2d$, leads to the following recursion relations (for $s = 1$)

$$\exp(\Delta') = \exp[\Delta - (\lambda - 1)D] \tag{11}$$

and

$$1 - \exp(-J') = (1 - e^{-\lambda J})^b\{1 - \exp[\Delta - (\lambda/d - 1)D]^{1-b}\}, \tag{12}$$

where $\lambda = b^{d-1}$ and from (10) $e^D = 1 + e^{-\Delta}$.

The analysis of these recursion relations, in the infinitesimal limit $b \to 1$, gives one fixed point at $\Delta = -\infty$ with $\exp(-J^*) = \frac{1}{2}$ for $d = 2$ and $\exp(-J^*) = 0.84$ for $d = 3$ with one relevant eigenvalue $y = 0.61$ for $d = 2$ and $y = 0.82$ for $d = 3$, confirming that random site–bond percolation is described by only one universality class. The phase diagram in the plane p_B, p_S is given in figure 2 for $d = 2$ and $d = 3$, in qualitative agreement with more accurate numerical results (see e.g. Agrawal *et al* 1979, Hermann and Stauffer 1981). We note that for $d = 3$ we find the desired feature $p_{Sc} < \frac{1}{2}$ for large enough values of p_B.

In conclusion, we have shown that in order to study site–bond percolation using the MKRG applied to the dilute s-state Potts model in the limit $s \to 1$, one has to treat the single-site interaction in a way that differs from the usual procedures, so that the physics underlying this problem is respected. The effect of this treatment in the general case of $K \neq 0$ (site–bond correlated percolation) and $s \neq 1$ (the dilute s-state Potts model) is under investigation.
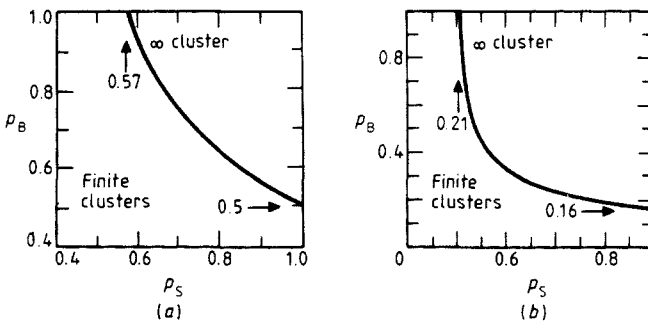

Figure 2. Phase diagram for random site percolation for (*a*) $d = 2$ and (*b*) $d = 3$.

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